Statistical Methods for Plant Biology

PBIO 3150/5150

Anirudh V. S. Ruhil

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The Voinovich School of Leadership and Public Affairs

Table of Contents

Probabilities

Probability

Definition

Numerical measure of the likelihood that an event will occur.
It is the proportion of times the event would occur if we repeated the random trial under identical conditions an infinite number of times.
By definition, $0 \leq P(Event) \leq 1$

Example

1. What proportion of the products in a production lot will be defective?
2. What is the probability of drawing ♠ from a single deck of cards?
3. What is the probability that a randomly selected Gliding Snake undulates at 1.38 Hz?
4. What is the probability a Freshman @ OU, chosen at random, has green eyes? What if this individual is a male?
5. What is the probability of a 15-29 year-old infected with the *Gondii* parasite ending up in a car accident?

Experiments (Random Trials)

An experiment is a process that generates well-defined outcomes
Each experimental outcome is a sample point
The sample space for an experiment is the set of all experimental outcomes
Events are assumed to be mutually exhaustive

Experiment	Outcomes	Sample Space
Coin Toss	Head, Tail	$S = \{H, T\}$
Sales Call	Sales, No Sale	$S = \{$Sales, No Sale$\}$
Roll a Dice	1,2,3,4,5,6	$S = \{1, 2, 3, 4, 5, 6\}$
Product Test	Defective; Not defective	$S = \{$Defective, Not Defective$\}$
Health Campaign	Behavior modified; Not	$S = \{$Modified; Not$\}$
Green Eyes	Yes; No	$S = \{$Yes; No$\}$
Infected	Has accident; Does not	$S = \{$Accident; No Accident$\}$

Random Variables

- A random variable is a numerical description of the outcome of an experiment.
- A discrete random variable assumes discrete values while a continuous random variable may assume any value in an interval or collection of intervals.

Example

Discrete Random Variable (x)	Possible Values
No. of defective iPhones	$0, 1, 2, 3, \cdots, 49$
Sex of car buyer	0 (Male); 1 (Female)
No. of Mountain Lions seen	$0, 1, 2, 3, \cdots, 419$
Gene Length (number of nucleotides)	$60 \leq x \leq 100,000$
Continuous Random Variable (x)	**Possible Values**
Spending per week	$0 \leq x \leq +\infty$
Travel times to CMH (minutes)	$55.3 \leq x \leq 118.5$
Undulation rates of Gliding Snakes	$0 \leq x \leq 1.9$
Petal Length of the virginica Iris (in cm)	$1 \leq x \leq 6.7$

Probability Distributions

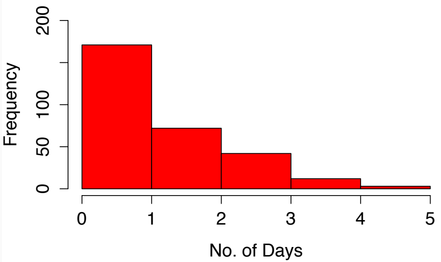
Probability Distributions

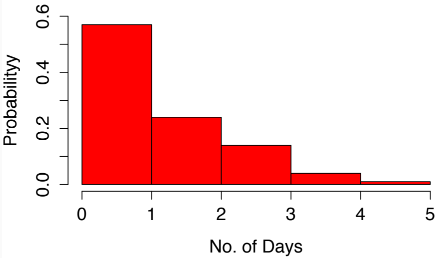
A probability distribution is a list of the probabilities of all mutually exclusive outcomes of an experiment

1. Discrete probability distributions
2. Continuous probability distributions

A probability distribution $f(Y)$ of a discrete random variable (Y) describes how probabilities are distributed over the values of the random variable. Discrete probability functions must meet two conditions:
(a) $f(Y) \geq 0$, and (b) $\sum f(Y) = 1$

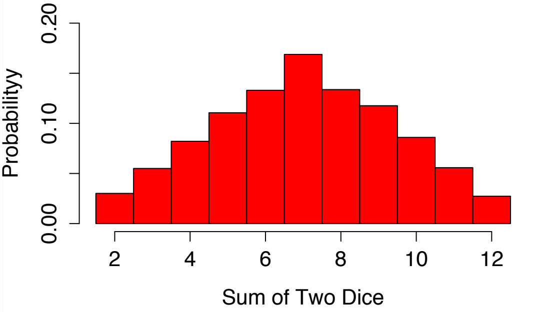
Mountain Lion Sightings (Y)	Days	$f(Y)$
0	54	$f(0) = 0.18$
1	117	$f(1) = 0.39$
2	72	$f(2) = 0.24$
3	42	$f(3) = 0.14$
4	12	$f(4) = 0.04$
5	3	$f(5) = 0.01$
Total	300	$\sum f(Y) = 1.00$


Frequency
0
50
100
200
0
1
2
3
4
5
No. of Days


Probabilityy
0.0
0.2
0.4
0.6
0
1
2
3
4
5
No. of Days

Rolling Two Dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12



Counting Rules

Multiple-step Experiments

1. Let us consider tossing two coins. $S = \{H,H\},\{H,T\},\{T,H\},\{T,T\}$
2. Generally, in a multi-step experiment with k sequential steps with n_1 outcomes in Step 1, n_2 outcomes in Step 2, n_3 outcomes in Step 3, ... and n_k outcomes in Step k, the total number of experimental outcomes is given by $(n_1)(n_2)(n_3)\cdots(n_k)$
3. For e.g., tossing two coins yields $(n_1)(n_2) = (2)(2) = 4$ outcomes
4. Likewise, tossing six coins yields $(n_1)(n_2)(n_3)(n_4)(n_5)(n_6) = (2)(2)(2)(2)(2)(2) = 64$ outcomes

Counting Rules (continued ...)

Another counting rule allows us to calculate the number of experimental outcomes when the experiment involves drawing n objects from a finite set of N objects

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where $N! = N(N-1)(N-2)\cdots(2)(1)$, and $n! = n(n-1)(n-2)\cdots(2)(1)$

Note that ! stands for "factorial"

1. $0! = 1$
2. $3! = 3 \times 2 \times 1 = 6$
3. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Example of Counting Rules in Action

1. Assume a quality control inspector at Apple's iPhone facility in Taiwan selects 2 iPhone 6 units for inspection out of a batch of 5 phones. In how many combinations can these 2 be selected? Applying the counting rule,

$$C_2^5 = \binom{5}{2} = \frac{5!}{2!(5-2)!}$$

$$= \frac{(5)(4)(3)(2)(1)}{(2)(1)(3)(2)(1)} = \frac{(5)(4)\cancel{(3)}\cancel{(2)}\cancel{(1)}}{(2)(1)\cancel{(3)}\cancel{(2)}\cancel{(1)}} = \frac{(5)(4)}{(2)(1)} = (5)(2) = 10$$

2. let the iPhones be labeled A, B, C, D, and E. Then the selections can be:

AB, AC, AD, AE, BC, BD, BE, CD, CE, DE

$S = \{AB, AC, AD, AE, BC, BD, BE, CD, CE, DE\}$

Example of Counting Rules in Action

1. The Ohio lottery randomly draws 6 integers from a group of 47 to draw the winner. How many winning combinations are possible?

$$C_6^{47} = \binom{47}{6} = \frac{47!}{6!(47-6)!}$$

$$= \frac{(47)(46)(45)(44)(43)(42)\cancel{(41)}\cancel{(40)}\cancel{(\ldots)}\cancel{(1)}}{(6)(5)(4)(3)(2)(1)\cancel{(41)}\cancel{(40)}\cancel{(\ldots)}\cancel{(1)}} = 10,737,573$$

2. If only one winner is possible and a buyer can only purchase one ticket each, what is the probability of buying the wnning ticket?

Counting Rules (continued ...)

A third counting rule is the counting rule for permutations that allows us to calculate the number of experimental outcomes when n objects are to be selected from a total of N objects in a particular order

1. $P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!}$
2. Let us assume 2 iPhone 6 units are to be drawn from 5 for quality control tests. In how many ways can we do this if the order matters?
3. $$P_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{(5)(4)(3)(2)(1)}{(3)(2)(1)} = \frac{(5)(4)\cancel{(3)}\cancel{(2)}\cancel{(1)}}{\cancel{(3)}\cancel{(2)}\cancel{(1)}} = 20$$
4. The 20 permutations are ... AB, BA, AC, CA, AD, DA, AE, EA, BC, CB, BD, DB, BE, EB, CD, DC, CE, EC, DE, and ED ... This is the Sample Space

Assigning Probabilities

Thus far we've seen how to use counting rules to establish the Sample Space. Now let us think about the probabilities associated with each outcome

We can assign probabilities to outcomes so long as we use two rules

1. For an event E_i, $0 \leq P(E_i) \leq 1$
2. Given n outcomes, $P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n) = 1$

For e.g., in tossing a fair coin, $P(H) = 0.5; P(T) = 0.5; \therefore P(H) + P(T) = 1$

Likewise, in tossing a fair dice, $P(1) = \frac{1}{6}; P(2) = \frac{1}{6}; \cdots P(6) = \frac{1}{6}$
Therefore, $P(1) + P(2) + \cdots + P(6) = 1$

This method of assigning probabilities is known as the classical method ... i.e., all outcomes are equally likely

Assigning Probabilities (Relative Frequency Method)

Assume a clerk in Obleness' X-ray unit tracks how many patients are on the waiting list at 9 AM on 20 successive days. These data are given below ...

No. Waiting	No. Days	P(E)
0	2	$P(0) = 2/20 = 0.10$
1	5	$P(1) = 5/20 = 0.25$
2	6	$P(2) = 6/20 = 0.30$
3	4	$P(3) = 4/20 = 0.20$
4	3	$P(4) = 3/20 = 0.15$
Total	20	1.00

Assigning Probabilities (Subjective Frequency Method)

In most situations where hard data are lacking, we rely on theories or then on our subjective beliefs to assign probabilities to likely outcomes …

Assume, for example, that a couple makes an offer on a house but each holds different probabilities of bid acceptance and rejection

Person	P(Acceptance)	P(Rejection)	Sum
Pat	0.80	0.20	1.00
Chris	0.60	0.40	1.00

Events and their Probabilities

An event is a collection of one or more sample points

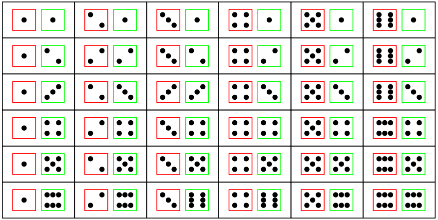
P(Event) is the sum of the probabilities of the sample point(s) in the event

2 dice are rolled and we are interested in sum of face values showing on the 2 dice

Possible outcomes: {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6) ... (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

In other words, $C_1^6 \times C_1^6 = (6)(6) = 36$

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12



. . . continued

1. What is P(value of 7)?

   $$P(7) = \frac{\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}}{36} = \frac{6}{36} = \frac{1}{6}$$

2. What is P(value ≥ 9)? $P(\geq 9) = \frac{10}{36} = \frac{5}{18}$
3. Will sum of dice show even values more than odd values?

   No because P(Odd) = P(Even) = $\frac{18}{36} = \frac{1}{2}$ for each
4. How did you assign probabilities? Classical: because each outcome has an identical probability of occurring

Complement of an Event

Definition

Given an event A, its complement (A^c) is defined as the event consisting of all sample points that do not belong to (i.e., are not in) event A .

$$
\begin{aligned}
P(A)+P(A^c) &= 1 \\
\therefore P(A) &= 1-P(A^c) \\
\therefore P(A^c) &= 1-P(A)
\end{aligned}
$$

- Toss a coin once: $P(H)=0.5; P(H^c)=1-P(H)=1-0.5=0.5$
- Roll two dice: $P(\text{value } \geq 9)=\frac{5}{18}$

  $P(\text{value } < 9)=1-P(\text{value } \geq 9)=1-\frac{5}{18}=\frac{13}{18}$

Mutually Exclusive Events

Definition

Two events are mutually exclusive if both cannot occur simultaneously. That is, if event A occurs then event B cannot occur, and vice-versa.

I toss a coin once. It can only come up Heads or Tails. Let, for example,

$$P(A) = Heads; P(B) = Tails$$
$$P(A \text{ and } B) = 0$$

I roll a dice once. Let, for example,

$$P(A) = \{1,3,5\}; P(B) = \{2,4,6\}$$
$$P(A \text{ and } B) = 0$$
$$\ldots \text{ because there is no overlap}$$

I roll a dice once. Let, for example,

$$P(A) = \{2,4,6\}; P(B) = \{1,4\}$$
$$P(A \text{ and } B) \neq 0$$
$$\ldots \text{ because there is an overlap}$$

The Addition Rule for Mutually Exclusive Events

The Addition Rule

Definition

For two mutually exclusive events, A and B, the probability that either A *or* B occurs is given by $P(A \text{ or } B) = P(A) + P(B)$. This rule also extends to more than 2 events so long as they are mutually exclusive.

A dice is rolled once. Let $A = \{3, 4, 5, 6\}$. What is $P(3 \text{ or more})$?

$$\begin{aligned} &= P(3) + P(4) + P(5) + P(6) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{4}{6} \\ &= \frac{2}{3} \end{aligned}$$

What is $P(\text{not rolling 3 or more})$? ... $1 - P(3 \text{ or more}) = 1 - \frac{2}{3} = \frac{1}{3}$

Addition Rule for Non-Mutually Exclusive Events

For non-mutually exclusive events we calculate the probability that event A or B occurs as $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Example

Assume on a typical day in a plant, of 50 workers 5 complete the work late, 6 assemble a defective product, and 2 both complete the work late and produce a defective product

1. Let L = Late completion. Then, $P(L) = \frac{5}{50} = 0.10$
2. Let D = Defective product. Then, $P(D) = \frac{6}{50} = 0.12$
3. $P(L \text{ and } D) = \frac{2}{50} = 0.04$
4. What is the probability that a randomly selected worker is either late or produces a defective product?
   $P(L \text{ or } D) = P(L) + P(D) - P(L \text{ and } D) = 0.10 + 0.12 - 0.04 = 0.18$

Independence and the Multiplication Rule

Independence and the Multiplication Rule

Definition

Two events, A and B, are independent events if P(A) is not influenced by whether event B has occurred or not (and vice-versa)

1. Rolling a 4, and rolling a 1 on a second roll of the same dice
2. Picking the Ace of Spades from a fair deck of 52 cards, replacing it, and then choosing the Ace of Spades again

If two events A and B are independent, then the probability that both A and B occur is given by $P(A \text{ and } B) = P(A) \times P(B)$

1. ... for ① above $P(A \text{ and } B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
2. ... for ② above $P(A \text{ and } B) = \frac{1}{52} \times \frac{1}{52} = \frac{1}{2704}$

Probability Trees

An Example

Some couples would like to have at least one child of each sex. The probability the first child is a boy is 0.512, which means the probability the first child is a girl is 0.488. If a couple have two children, what is the probability of:

1. Two boys? $= 0.512 \times 0.512 = 0.262144$
2. Two girls? $= 0.488 \times 0.488 = 0.238144$
3. One boy and One girl? This could happen in two ways:

   A = First is a Boy and second is a girl; B = first is a girl and second is boy

   $P(A \text{ or } B) = (0.512 \times 0.488) + (0.488 \times 0.512)$

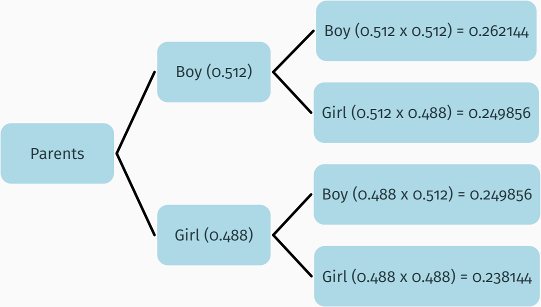
   $= 0.249856 + 0.249856 = 0.499712$
4. Both are of the same sex? This could happen in two ways:

   A = Two boys; B = Two girls

   $P(A \text{ or } B) = 0.262144 + 0.238144 = 0.500288$
5. $P(\text{at least 1 girl})? = 1 - P(2 \text{ boys}) = 1 - 0.262144 = 0.737856$
6. $P(\text{at least 1 boy})? = 1 - P(2 \text{ girls}) = 1 - 0.238144 = 0.761856$

Probability Trees

Trees are handy ways to depict sequential events and their probabilities



Dependent Events

Conditional Probability

Definition

Conditional probability of an event is the probability of an event occurring given that a condition is met (i.e., some other event is known to have occurred). Conditional probabilities are denoted as $P(A|B)$ (i.e., the probability of A given that B has occurred)

$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$ while $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$

Let B be an event of getting a perfect square when a dice is rolled. Let A be the event that the number on the dice is an odd number. What is $P(B|A)$?

Sample Space $S = \{1,2,3,4,5,6\}$; $A = \{1,3,5\}$; $B = \{1,4\}$

Then, $P(A \text{ and } B) = \frac{1}{6}; P(A) = \frac{1}{2}; P(B) = \frac{1}{3}$

and so $P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{6} \times \frac{2}{1} = \frac{2}{6} = \frac{1}{3}$

The LAPD Example

Action	Men (M)	Women (W)	Total
Promoted (A)	288	36	324
Not Promoted (A^c)	672	204	876
Total	960	240	1200

Action	Men (M)	Women (W)	Total
Promoted (A)	$P(A \text{ and } M) = 0.24$	$P(A \text{ and } W) = 0.03$	$P(A) = 0.27$
Not Promoted (A^c)	$P(A^c \text{ and } M) = 0.56$	$P(A^c \text{ and } W) = 0.17$	$P(A^c) = 0.73$
Total	$P(M) = 0.80$	$P(W) = 0.20$	1.00

1. If Men and Women had the same probability of being promoted each group should have $P(A) = \dfrac{324}{1200} = 0.27$
2. What is the probability that an officer is promoted *given that* the officer is a man? $P(A|M) = \dfrac{P(A \text{ and } M)}{P(M)} = \dfrac{288/1200}{960/1200} = \dfrac{288}{960} = 0.30$
3. What is the probability of being promoted *given that* the officer is a woman? $P(A|W) = \dfrac{P(A \text{ and } W)}{P(W)} = \dfrac{36/1200}{240/1200} = \dfrac{36}{240} = 0.15$

Dependent Events

Many events are not independent of one another; the odds of event A change if event B has occurred (and vice versa).

The jewel wasp *Nasonia vitripennis* is a parasite laying its eggs on the pupae of flies. The larval *Nasonia* hatch inside the pupal case, feed on the live host, and grow until they emerge as adults from the now dead, emaciated host. Emerging males and females, possibly brother and sister, mate on the spot.

Nasonia females have a remarkable ability to manipulate the sex of the eggs that they lay; if they fertilize the egg with stored sperm the offspring will be a female. When a female finds a fresh host (i.e., not parasitized), she lays mainly female eggs and a few sons needed to fertilize all her daughters. If the female finds the host to be parasitized she produces a higher proportion of sons. Thus the state of the host (parasitized or not) and the sex of an egg are dependent events.

Let the probability a host already has eggs be = 0.20. If it is a fresh host, the female lays a male egg with a probability of 0.05 and a female egg with a probability of 0.95. If the host already has eggs the female lays a male egg with a robability of 0.90 and a female egg with a probability of 0.10.
What is the probability that an egg, chosen at random, is male?

The Nasonia Example again

The Law of Total Probability stipulates that the total probability of an event A occurring is given by $P(A) = [P(A|B) \times P(B)] + [P(A|B^c) \times P(B^c)]$

The Multiplication Rule: The probability of both of two events occurring is given by $P(A \text{ and } B) = P(A) \times P(B|A)$.

If the two events are *independent* then we know that $P(A \text{ and } B) = P(A) \times P(B)$.

Example

What is $P(\text{egg is male})$? We don't know if the host is parasitized so ...

$P(\text{egg is male}) = P(\text{host parasitized}) \times P(\text{egg is male} \mid \text{host parasitized})$
+ $P(\text{host not parasitized}) \times P(\text{egg is male} \mid \text{host is not parasitized})$

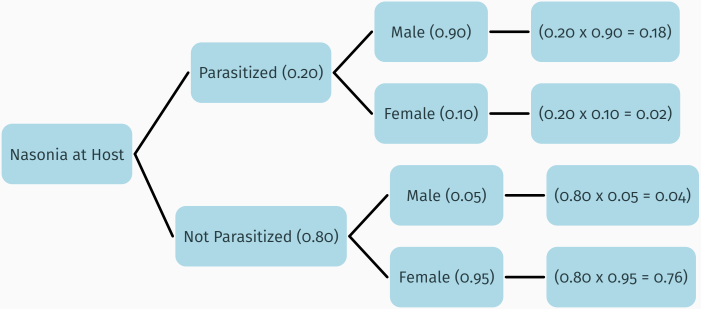
$P(\text{egg is male}) = (0.20 \times 0.90) + (0.80 \times 0.05) = 0.22$

What is $P(\text{egg is female})$? We don't know if the host is parasitized so ...

$P(\text{egg is female}) = P(\text{host parasitized}) \times P(\text{egg is female} \mid \text{host parasitized})$
+ $P(\text{host not parasitized}) \times P(\text{egg is female} \mid \text{host is not parasitized})$

$P(\text{egg is female}) = (0.20 \times 0.10) + (0.80 \times 0.95) = 0.78$

the Decision tree



Bayes' Theorem

Bayes' Theorem stipulates that $P(A|B) = \dfrac{P(B|A) \times P(A)}{P(B)}$.

In fact ... $P(A|B) = \dfrac{P(B|A) \times P(A)}{[P(B|A) \times P(A)] + [P(A|B^c) \times P(B^c)]}$

So what?

Example

"1% of women at age forty who participate in routine screening have breast cancer. 80% of women with breast cancer will get positive mammographies. 9.6% of women without breast cancer will also get positive mammographies.

A 40-year old woman had a positive mammography in a routine screening.

What is the probability that she actually has breast cancer?"

	Cancer (B)	No Cancer (B^c)		
$+ (A)$	0.800	0.096		
$- (A^c)$	0.200	0.904		
	(B)	(B^c)	Total	P(?)
$+ (A)$	80.00	950.40	1,030.40	$P(B\|+) = \frac{80.00}{1030.40} = 0.0776$
$- (A^c)$	20.00	8,949.60	8,969.60	$P(B\|-) = \frac{20.00}{8969.60} = 0.0022$
Total	100.00	9,900.00	10,000.00	

What is $P(\text{Breast Cancer} \mid +\text{Mammography})$, i.e., $P(B|A)$?

$$P(B|A) = \frac{P(A|B) \times P(B)}{[P(A|B) \times P(B)] + [P(A|B^c) \times P(B^c)]}$$

$$= \frac{0.800 \times 0.01}{[0.800 \times 0.01] + [0.096 \times 0.990]} = \frac{0.008}{[0.008 + 0.09504]} = \frac{0.008}{0.10304} = 0.0776$$

The Simpler Version of Bayes' Theorem

$P(B|A) = \frac{P(A|B) \times P(B)}{P(A)}$

What is $P(A|B)$ … the probability that a woman has Breast Cancer and gets a positive mammography $= 0.80$

What is $P(B)$? … the probability of breast cancer $= 0.01$

What is $P(A)$? … This is the probability of getting a positive mammography, which can happen

1. *when you have Breast Cancer* … (0.80×0.01), or
2. *when you don't have Breast Cancer* … (0.096×0.99)

$$\therefore P(A) = (0.80 \times 0.01) + (0.096 \times 0.99) = 0.10304$$

$$\therefore P(B|A) = \frac{P(A|B) \times P(B)}{P(A)}$$

$$= \frac{0.80 \times 0.01}{0.10304} = \frac{0.008}{0.10304} = 0.07763975$$

Detection of Downs Syndrome

	DS	No DS	Total	P(?)
$+Test$	600	49,950	50,550	$P(DS\|+Test) = \frac{600}{50,550} = 0.0118$
$-Test$	400	949,050	949,450	$P(DS\|-Test) = \frac{400}{949,450} = 0.0004$
Total	1,000	999,000	1,000,000	

What is $P(DS|+Test)$?

We know that $P(DS|+Test) = \dfrac{P(+Test|DS) \times P(DS)}{P(+Test)}$

We know that $P(+Test|DS) = 0.60$ and that $P(DS) = 0.001$ so we only need calculate $P(+Test)$ via the Law of Total Probability ...

$$P(+\text{Test}) = [P(+Test|DS) \times P(DS)] + [P(+Test|DS^c) \times P(DS^c)]$$

$$= [0.60 \times 0.001] + [0.05 \times 0.999] = 0.0006 + 0.04995 = 0.05055$$

Now we can wrap up our calculations ...

$$P(DS|+Test) = \frac{P(+Test|DS) \times P(DS)}{P(+Test)} = \frac{0.60 \times 0.001}{0.05055} = \frac{0.0006}{0.05055}$$

$$\therefore P(DS|+Test) = 0.01186944$$

Extending Bayes' Rule

An emergency locator transmitter (ELT) transmits a signal in the case of a crash. The Altigauge Manufacturing Company makes 80% of the ELTs, the Bryant Company makes 15% of them, and the Chartair Company makes the other 5%. The ELTs made by Altigauge have a 4% rate of defects, the Bryant ELTs have a 6% rate of defects, and the Chartair ELTs have a 9% rate of defects. What is $P(Altigauge|Defective)$?

	D	Not D	Total	P(?)
Altigauge	320	7,680	8,000	$P(A\|D) = \frac{320}{455} = 0.7032967$
Bryant	90	1,410	1,500	$P(B\|D) = \frac{90}{455} = 0.1978022$
Chartair	45	455	500	$P(C\|D) = \frac{45}{455} = 0.0989011$
Total	455	9,545	10,000	

$$
\begin{aligned}
P(A|D) &= \frac{P(D|A) \times P(A)}{[P(D|A) \times P(A)] + [P(D|B) \times P(B)] + [P(D|C) \times P(C)]} \\
&= \frac{0.80 \times 0.04}{[0.80 \times 0.04] + [0.15 \times 0.06] + [0.05 \times 0.09]} \\
&= \frac{0.0320}{0.0320 + 0.0090 + 0.0045} = \frac{0.0320}{0.0455} = 0.7032967
\end{aligned}
$$

Probability Redux

Addition Rule

1. Non-Mutually Exclusive Events: $P(\text{A or B}) = P(A) + P(B) - P(\text{A and B})$
2. Mutually Exclusive Events: $P(\text{A or B}) = P(A) + P(B)$

Multiplication Rule

1. Independent Events: $P(\text{A and B}) = P(A) \times P(B)$
2. Dependent Events: $P(\text{A and B}) = P(A) \times P(B|A)$
   $P(\text{B and A}) = P(B) \times P(A|B)$

A and B are mutually exclusive if $P(\text{A and B}) = 0$

A and B are independent if $P(A|B) = P(A)$; $P(B|A) = P(B)$

Total Probability: $P(B) = P(A) \times P(B|A) + P(A^c) \times P(B|A^c)$

Bayes' Rule: $P(A|B) = \dfrac{P(A) \times P(B|A)}{P(B)}$; and $P(B|A) = \dfrac{P(B) \times P(A|B)}{P(A)}$